

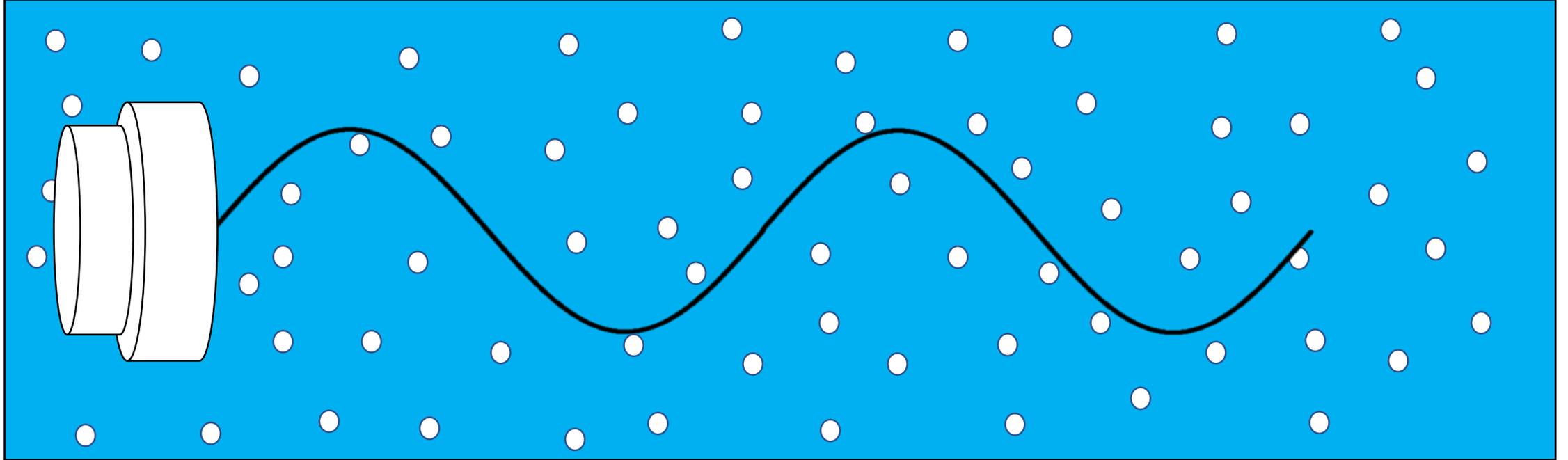
2018年10月18日 9:00~ 9:30
京都大学数理解析研究所

気泡を含む液体中の弱非線形波動 における粘性と熱伝導性の効果

○亀井 陸史 (筑波大・理工4年)

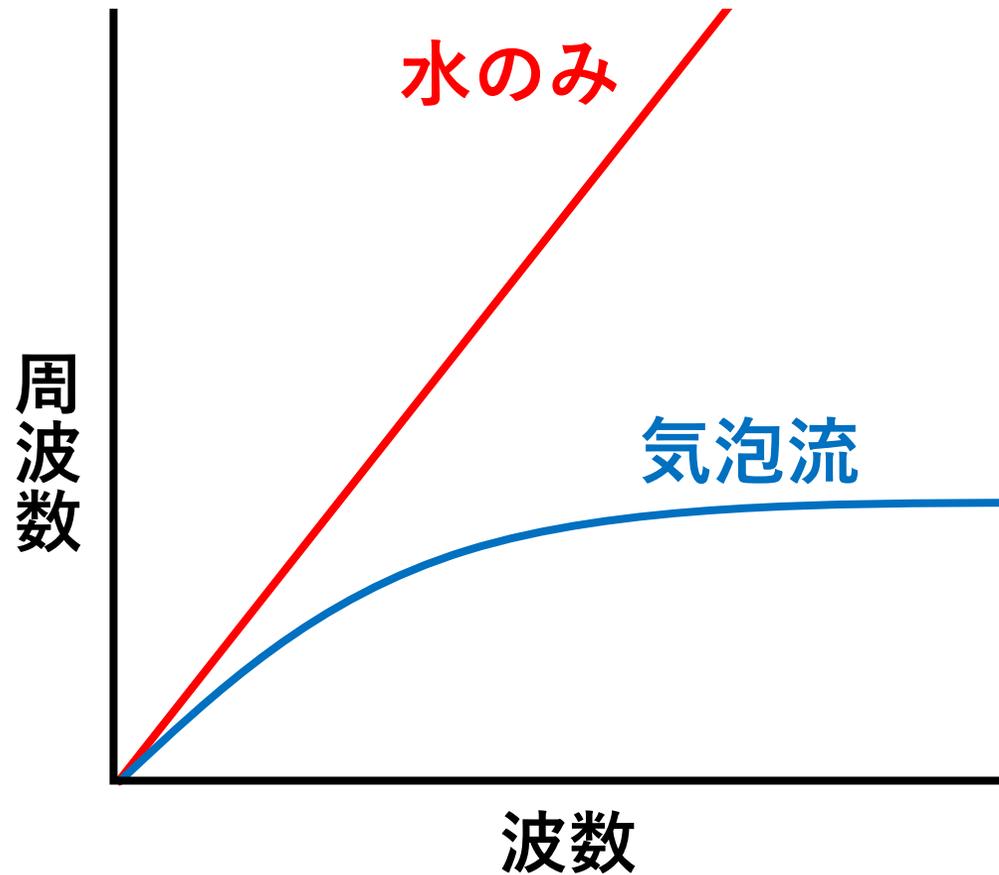
金川 哲也 (筑波大・システム情報)

問題



- 多数の気泡を含む液体中における音波
- 気泡は水より強い弾性をもつ
 - ➡ 音波の入射により激しく振動
- 弱非線形伝播の**分散性**に着目(van Wijngaarden, 1968)

線形分散関係 (e.g., van Wijngaarden, 1968)



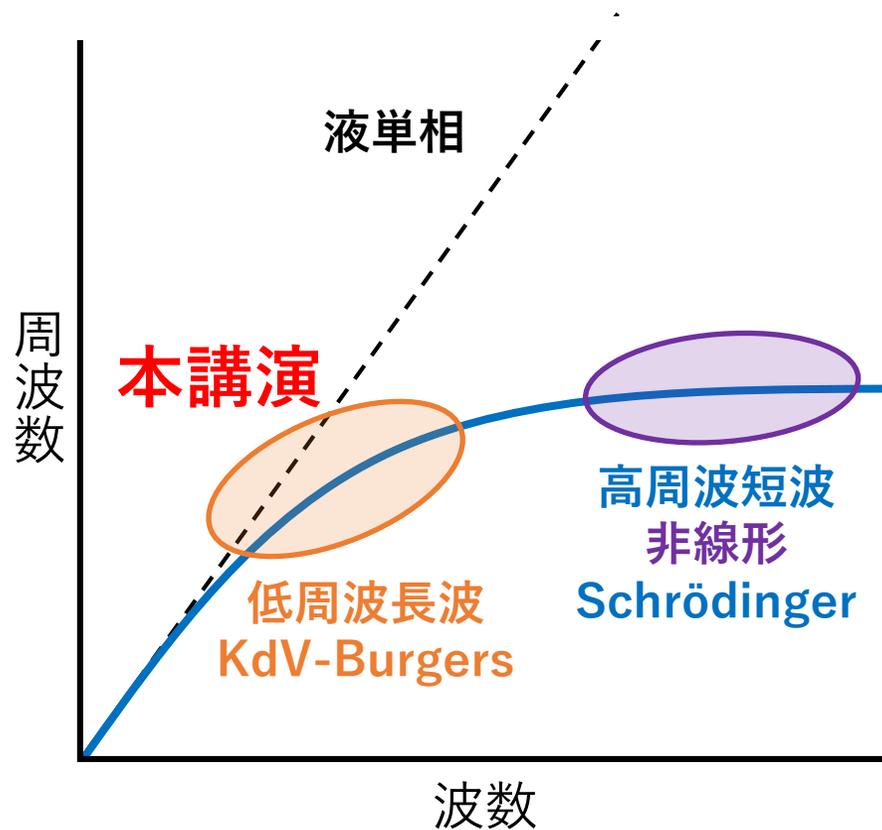
音速

気泡流
約150 m/s

<<

水
1500 m/s

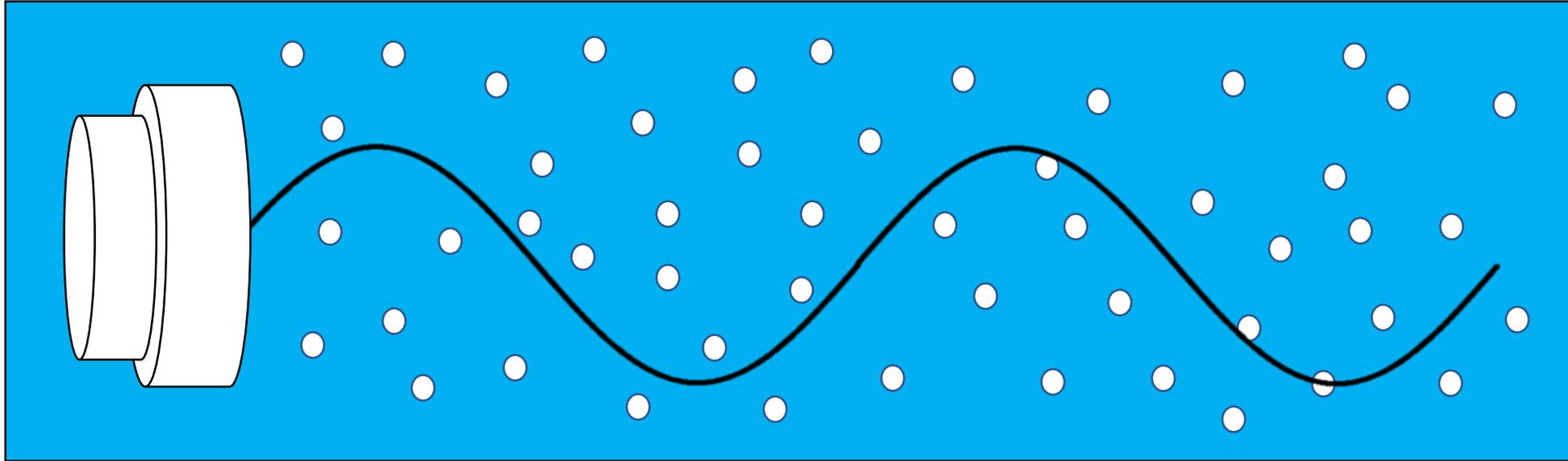
先行研究(金川, 矢野, 渡部, 藤川 J.Fluid Sci.Technol.,2010)



先行研究は熱伝導性と粘性を無視

熱伝導性と粘性を含む基礎方程式系を用いた
KdV-Burgers方程式の導出

問題設定



問題設定

- 波は**平面進行波**
- 液相および気泡は**初期静止**している
- 気泡は**球形**で一様に分布
- 気液界面で**相変化は起こらない**

基礎方程式系

基礎方程式系(保存則(e.g.,Ishii, 1977))

➤質量保存式

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial \rho^* u^*}{\partial x^*} = 0$$

➤運動量保存式

$$\frac{\partial \rho^* u^*}{\partial t^*} + \frac{\partial \rho^* u^{*2}}{\partial x^*} + \frac{\partial p_L^*}{\partial x^*} - \frac{4}{3} \mu^* \frac{\partial^2 u^*}{\partial x^{*2}} = 0$$

➤エネルギー保存式

$$\rho^* c_p^* \frac{DT^*}{Dt^*} = \beta^* T^* \frac{Dp_L^*}{Dt^*} + \frac{4}{3} \mu^* \left(\frac{\partial u^*}{\partial x^*} \right)^2 + \lambda^* \frac{\partial^2 T^*}{\partial x^{*2}}$$

*は全て有次元数
 β^* : 体積膨張係数
 μ^* : 粘性係数
 λ^* : 熱伝導率
 c_p^* : 定圧比熱

液相全体の粘性を考慮

熱伝導性を考慮

基礎方程式系

➤ ボイド率の定義式

$$\alpha \equiv \frac{4}{3}\pi R^{*3} N^*$$

➤ 数密度の保存式

$$\frac{\partial N^*}{\partial t^*} + \frac{\partial N^* u^*}{\partial x^*} = 0$$

➤ 密度の定義式

$$\rho^* = (1 - \alpha)\rho_L^*$$

*は全て有次元数

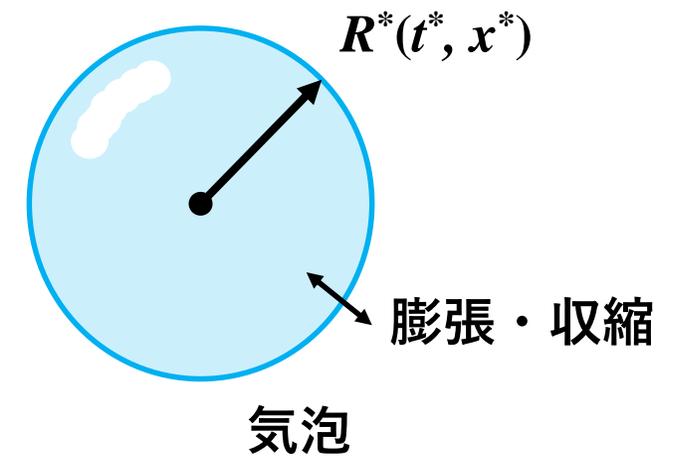
t^* : 時間 x^* : 座標

ρ^* : 密度 u^* : 速度

N^* : 気泡の数密度

気泡の運動方程式(Keller, 1956)

$$\left(1 - \frac{1}{c_{L0}^*} \frac{DR^*}{Dt^*}\right) R^* \frac{D^2 R^*}{Dt^{*2}} + \frac{3}{2} \left(1 - \frac{1}{3c_{L0}^*} \frac{DR^*}{Dt^*}\right) \left(\frac{DR^*}{Dt^*}\right)^2$$
$$= \left(1 + \frac{1}{c_{L0}^*} \frac{DR^*}{Dt^*}\right) \frac{P^*}{\rho_{L0}^*} + \frac{R^*}{\rho_{L0}^* c_{L0}^*} \frac{D}{Dt^*} (p_L^* + P^*)$$



構成式

- 液相におけるTait型の状態方程式

$$p_L^* = p_{L0}^* + \frac{\rho_{L0}^* c_{L0}^{*2}}{n} \left[\left(\frac{\rho_L^*}{\rho_{L0}^*} \right)^n - 1 \right]$$

- 気相における状態方程式

$$\frac{p_G^*}{p_{G0}^*} = \frac{\rho_G^*}{\rho_{G0}^*} \frac{T^*}{T_0^*}$$

$$\frac{p_G^*}{p_{G0}^*} = \left(\frac{\rho_G^*}{\rho_{G0}^*} \right)^\gamma$$

- 気泡内気体の質量保存式

$$\frac{\rho_G^*}{\rho_{G0}^*} = \left(\frac{R_0^*}{R^*} \right)^3$$

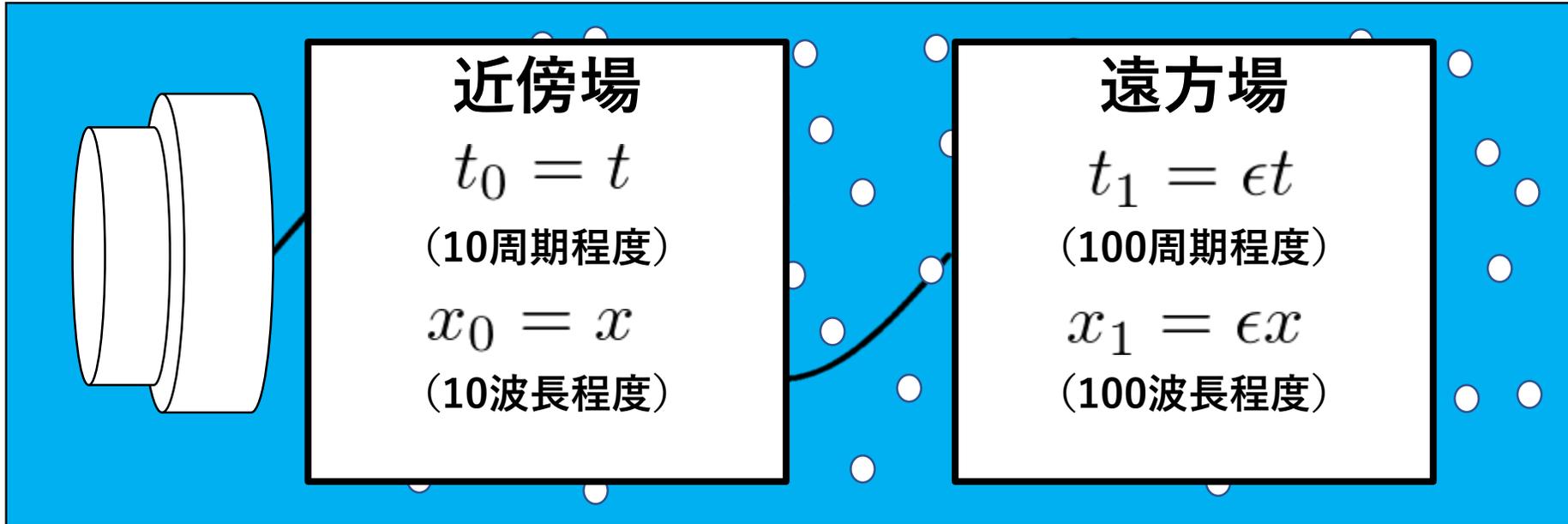
- 気液界面における法線方向の応力のつり合いの式

$$p_G^* - (p_L^* + P^*) = \frac{2\sigma^*}{R^*} + \frac{4\mu^*}{R^*} \frac{DR^*}{Dt^*}$$

解析手法

多重尺度法 (e.g., Jeffrey & Kawahara, 1982)

- 波の無次元振幅 $\epsilon (\ll 1)$ を導入し, 近傍場と遠方場を表す



- 摂動展開 (一部)

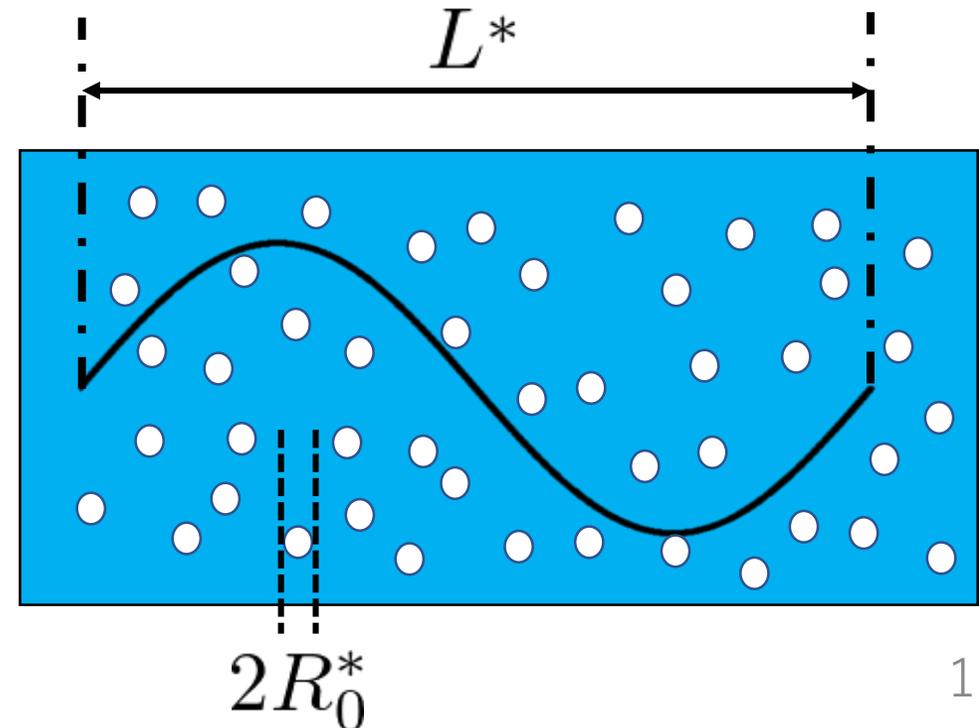
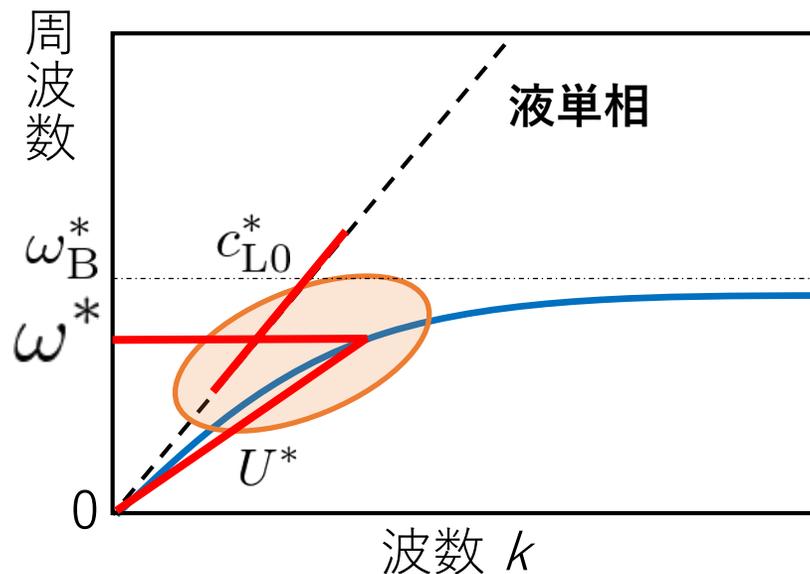
$$\frac{R^*}{R_0^*} = 1 + \epsilon R_1 + \epsilon^2 R_2 + O(\epsilon^3)$$

$$\frac{T^*}{T_0^*} = 1 + \epsilon T_1 + \epsilon^2 T_2 + O(\epsilon^3)$$

パラメータスケーリング (金川ら, 2010)

- 波の位相速度 U^* ，波長 L^* ，入射周波数 ω^* に対して，無次元数のスケーリングを波の無次元振幅 ϵ を用いて決定

$$\omega^* \ll \omega_B^* \quad U^* \ll c_{L0}^* \quad R_0^* \ll L^*$$



保存式における無次元数

- 運動量保存式

$$\frac{\partial}{\partial t} [(1 - \alpha) \rho_L u] + \frac{\partial}{\partial x} [(1 - \alpha) \rho_L u^2] + \frac{\partial p_L}{\partial x} - \frac{4}{3} \frac{\mu^*}{\rho_{L0}^* U^* L^*} \frac{\partial^2 u}{\partial x^2} = 0$$

$\equiv \mu\epsilon$

- エネルギー保存式

$$(1 - \alpha) \rho_{L0} \frac{DT}{Dt} = \frac{\beta^* U^{*2}}{c_p^*} T \frac{Dp_L}{Dt} + \frac{\lambda^*}{\rho_{L0}^* U^* L^* c_p^*} \frac{\partial^2 T}{\partial x^2} + \frac{U^{*2}}{c_p^* T_0^*} \frac{4}{3} \frac{\mu^*}{\rho_{L0}^* U^* L^*} \left(\frac{\partial u}{\partial x} \right)^2$$

$\equiv \delta$ $\equiv \lambda\epsilon$ $\equiv \eta$ $\equiv \mu\epsilon$

結果

基礎方程式系

➤ 数密度の保存式

$$\frac{\partial N^*}{\partial t^*} + \frac{\partial N^* u^*}{\partial x^*} = 0$$

➤ 質量保存式

$$\frac{\partial \rho^*}{\partial t^*} + \frac{\partial \rho^* u^*}{\partial x^*} = 0$$

➤ 運動量保存式

$$\frac{\partial \rho^* u^*}{\partial t^*} + \frac{\partial \rho^* u^{*2}}{\partial x^*} + \frac{\partial p_L^*}{\partial x^*} - \frac{4}{3} \mu^* \frac{\partial^2 u^*}{\partial x^{*2}} = 0$$

➤ 気泡の運動方程式

$$\begin{aligned} \left(1 - \frac{1}{c_{L0}^*} \frac{DR^*}{Dt^*}\right) R^* \frac{D^2 R^*}{Dt^{*2}} + \frac{3}{2} \left(1 - \frac{1}{3c_{L0}^*} \frac{DR^*}{Dt^*}\right) \left(\frac{DR^*}{Dt^*}\right)^2 \\ = \left(1 + \frac{1}{c_{L0}^*} \frac{DR^*}{Dt^*}\right) \frac{P^*}{\rho_{L0}^*} + \frac{R^*}{\rho_{L0}^* c_{L0}^*} \frac{D}{Dt^*} (p_L^* + P^*) \end{aligned}$$

➤ エネルギー保存式

$$\rho^* c_p^* \frac{DT^*}{Dt^*} = \beta^* T^* \frac{Dp_L^*}{Dt^*} + \frac{4}{3} \mu^* \left(\frac{\partial u^*}{\partial x^*}\right)^2 + \lambda^* \frac{\partial^2 T^*}{\partial x^{*2}}$$

$O(\epsilon)$: 線形化された基礎方程式系

$$\frac{\partial \alpha_1}{\partial t_0} - 3 \frac{\partial R_1}{\partial t_0} + \frac{\partial u_1}{\partial x_0} = 0$$

$$\alpha_0 \frac{\partial \alpha_1}{\partial t_0} - (1 - \alpha_0) \frac{\partial u_1}{\partial x_0} = 0$$

$$(1 - \alpha_0) \frac{\partial u_1}{\partial t_0} + \frac{\partial p_{L1}}{\partial x_0} = 0$$

➤ 気泡の運動方程式 $\left[3(\gamma - 1)p_{G0} - \frac{\Delta^2}{\Omega^2} \right] R_1 + p_{G0}T_1 - p_{L1} = 0$

➤ エネルギー保存式 $(1 - \alpha_0) \frac{\partial T_1}{\partial t_0} - \delta \frac{\partial p_{L1}}{\partial t_0} = 0$

本研究で**新出**

$O(\epsilon)$: 線形波動方程式

●先行研究

$$\frac{\partial^2 R_1}{\partial t_0^2} - \frac{\Delta^2 / \Omega^2}{3\alpha_0(1 - \alpha_0)} \frac{\partial^2 R_1}{\partial x_0^2} = 0$$

●本解析

$$\frac{\partial^2 R_1}{\partial t_0^2} - \frac{\frac{\Delta^2}{\Omega^2} \boxed{- 3(\gamma - 1)p_{G0}}}{3\alpha_0(1 - \alpha_0 \boxed{- \delta p_{G0}})} \frac{\partial^2 R_1}{\partial x_0^2} = 0$$

音速が熱の影響を受ける

$O(\epsilon^2)$: KdV-Burgers方程式

$$\frac{\partial^2 R_2}{\partial t_0^2} - \frac{\frac{\Delta^2}{\Omega^2} - 3(\gamma - 1)p_{G0}}{3\alpha_0(1 - \alpha_0 - \delta p_{G0})} \frac{\partial^2 R_2}{\partial x_0^2} = \boxed{K}$$

非同次項



可解条件

$$\frac{\partial R_1}{\partial \tau} + \boxed{\Pi_1 \frac{\partial^2 R_1}{\partial \xi^2}} + \boxed{\Pi_2 \frac{\partial^3 R_1}{\partial \xi^3}} + \boxed{\Pi_3 R_1 \frac{\partial R_1}{\partial \xi}} = 0$$

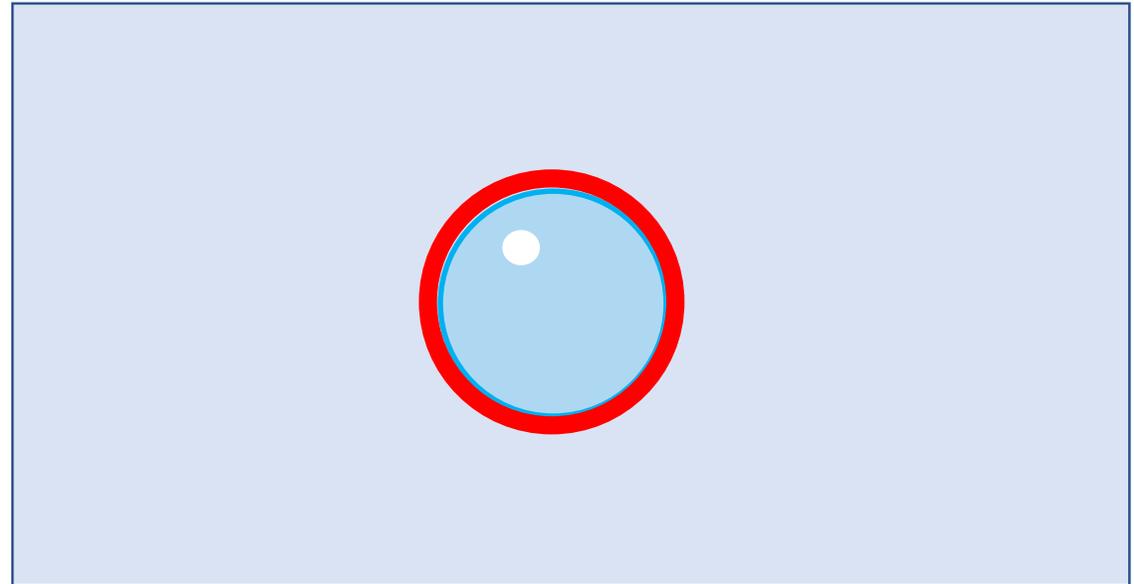
散逸項 分散項 非線形項

散逸係数と粘性・熱伝導の影響

●先行研究

$$4\mu + \frac{V\Delta^3}{\Omega^2}$$

音響放射



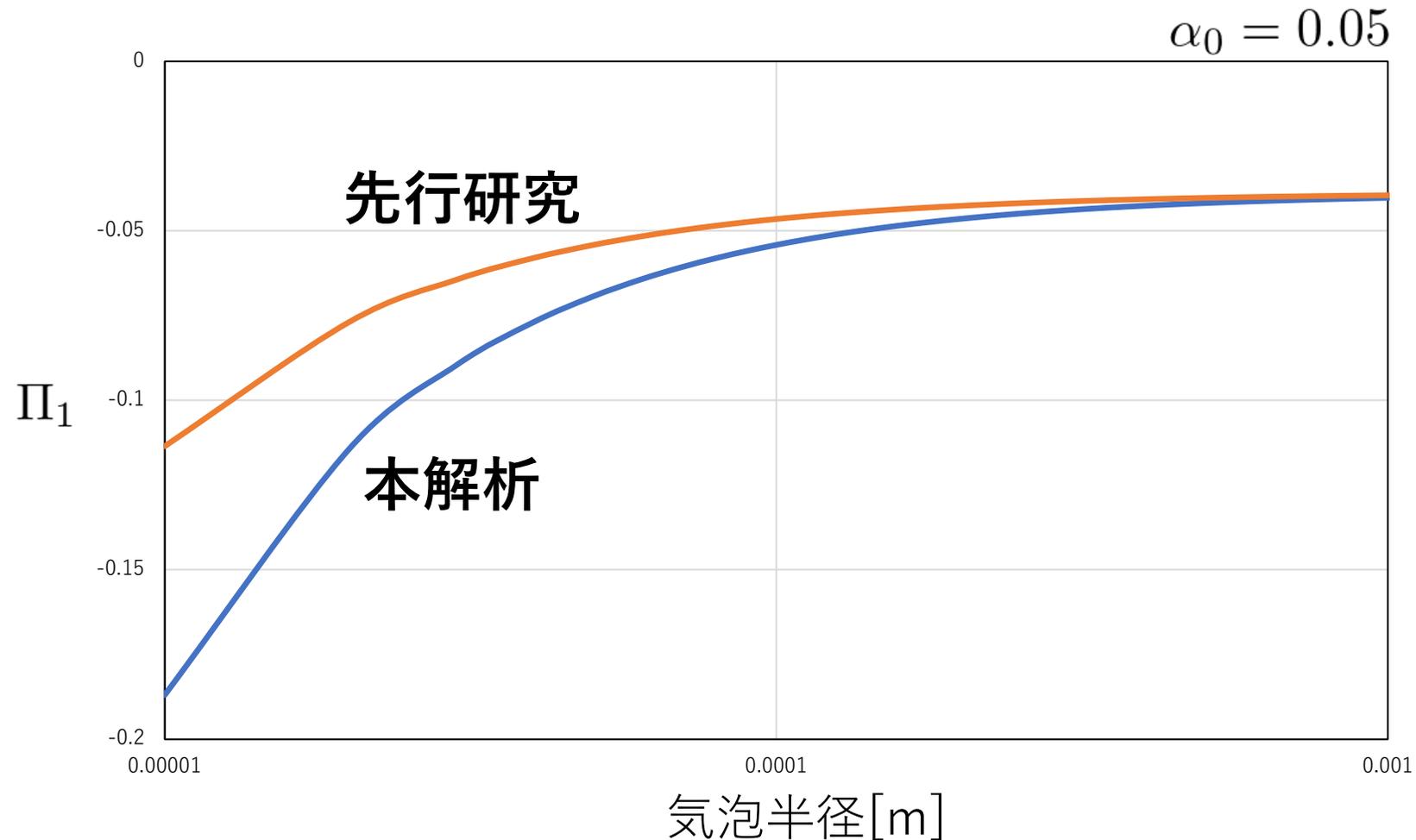
●本解析

$$4\mu + [4\mu + 3\alpha_0(1 - \alpha_0)V\Delta] \left(1 + \frac{\delta p_{G0}}{1 - \alpha_0 - \delta p_{G0}} \right) + 3\alpha_0\lambda \frac{\delta p_{G0}}{1 - \alpha_0 - \delta p_{G0}}$$

粘性 音響放射 エネルギー保存式由来 熱伝導

散逸係数と粘性・熱伝導の影響

- 粘性・熱伝導の組込みにより先行研究より**散逸性が強くなる**



分散係数・非線形係数と粘性・熱伝導

●先行研究

$$\Pi_2 = \frac{\Delta^2}{6\alpha_0(1 - \alpha_0)}$$

$$\Pi_3 = - \left[2 + \frac{\gamma(3\gamma - 1)}{2\alpha_0(1 - \alpha_0)} p_{G0} \right]$$

●本解析

$$\Pi_2 = \frac{\Delta^2}{6\alpha_0(1 - \alpha_0 - \delta p_{G0})}$$

$$\Pi_3 = - \left\{ 2 + \frac{2 + 3\alpha_0\delta[2 + \alpha_0(1 - \delta)]}{2\alpha_0(1 - \alpha_0 - \delta p_{G0})} p_{G0} \right\}$$

エネルギー保存式の効果

まとめ

- 目的

熱伝導性と粘性を含む基礎方程式系を用いた低周波長波における
KdV-Burgers方程式の導出

- 結果

- KdV-Burgers方程式を導いた

$$\frac{\partial R_1}{\partial \tau} + \Pi_1 \frac{\partial^2 R_1}{\partial \xi^2} + \Pi_2 \frac{\partial^3 R_1}{\partial \xi^3} + \Pi_3 R_1 \frac{\partial R_1}{\partial \xi} = 0$$

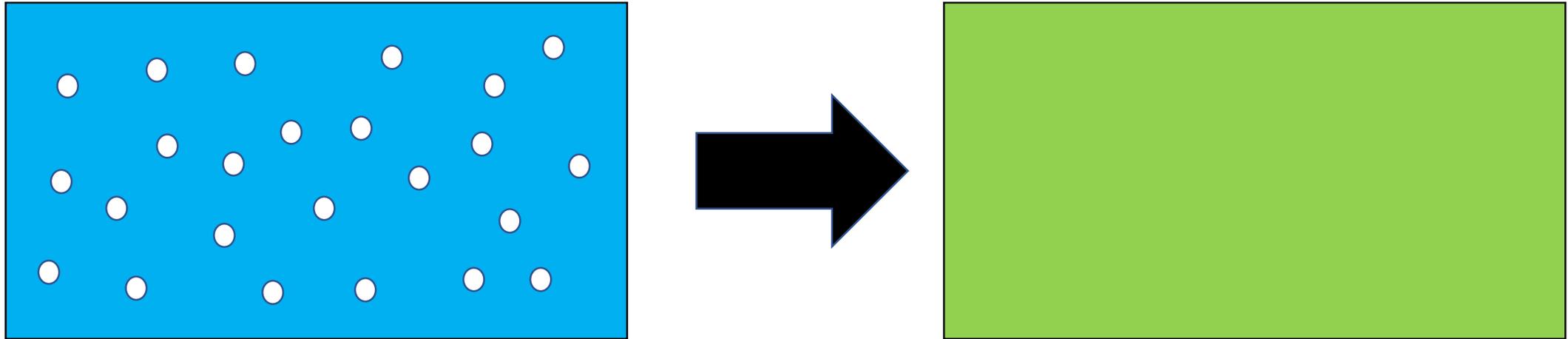
- 各係数に熱伝導性・粘性が組み込まれた

- 展望

- 得られたKdV-Burgers方程式を数値的に解いて波形を考察する

ご清聴ありがとうございました

混合体モデルとは？ (片岡, 1991)



- 実際は：界面を持つ
- 気泡流全体を1つの混合体とみなして平均化

➤ 密度の定義式

$$\rho^* = (1 - \alpha)\rho_L^*$$

$$\begin{aligned}\rho^* &= (1 - \alpha)\rho_L^* + \alpha\rho_G^* \\ &= (1 - \alpha)\rho_L^*\end{aligned}$$

$$\frac{\rho_G^*}{\rho_L^*} \equiv O(\epsilon^3)$$

- 気泡の固有周波数

$$\omega_B^* = \sqrt{\frac{3\gamma(p_{L0}^* + 2\sigma^*/R_0^*) - 2\sigma^*/R_0^*}{\rho_{L0}^* R_0^{*2}}}$$

- 固有周波数の例

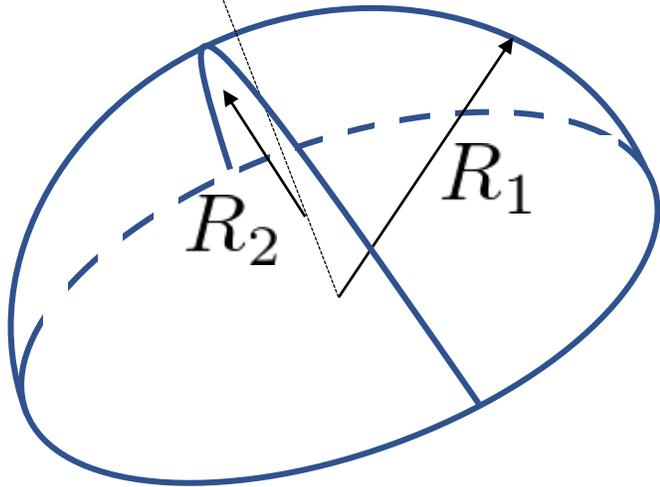
$$p_{L0}^* = 101325[\text{Pa}], \quad \sigma^* = 0.0728[\text{N/m}], \quad \rho_{L0}^* = 998[\text{kg/m}^3], \quad \gamma = 1$$

$$R_0^* = 1 \text{ mm} \Rightarrow \omega_B^* \simeq 10 \text{ [kHz]}$$

$$R_0^* = 10 \text{ }\mu\text{m} \Rightarrow \omega_B^* \simeq 1 \text{ [MHz]}$$

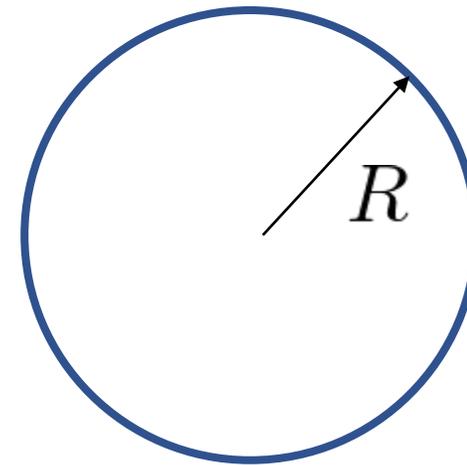
$$p_G^* - (p_L^* + P^*) = \frac{2\sigma^*}{R^*} + \frac{4\mu^*}{R^*} \frac{DR^*}{Dt^*}$$

表面張力による
気泡内外の圧力差



Young-Laplaceの式

$$\Delta P = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$



$$R_1 = R_2 = R$$

$$\Delta P = \frac{2\sigma}{R}$$

- **ボイド率**

$$\frac{\alpha}{\alpha_0} = 1 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + O(\epsilon^3)$$

- **流速**

$$\frac{u^*}{U^*} = \epsilon u_1 + \epsilon^2 u_2 + O(\epsilon^3)$$

- **密度**

$$\rho_L = \frac{\rho_L^*}{\rho_{L0}^*} = 1 + \epsilon^2 \rho_{L1} + O(\epsilon^3)$$

- **圧力**

$$p_L = \frac{p_L^*}{\rho_{L0}^* U^{*2}} = p_{L0} + \epsilon p_{L1} + \epsilon^2 p_{L2} + O(\epsilon^3)$$

• λ のスケーリング

$$\frac{\lambda^*}{\rho_{L0}^* U^* L^* c_p^*} = \frac{\mu^*}{\rho_{L0}^* U^* L^*} \frac{\lambda^*}{\mu^* c_p^*} = \frac{\mu}{\text{Pr}} \epsilon$$

$\equiv \mu\epsilon$ 常温で約7

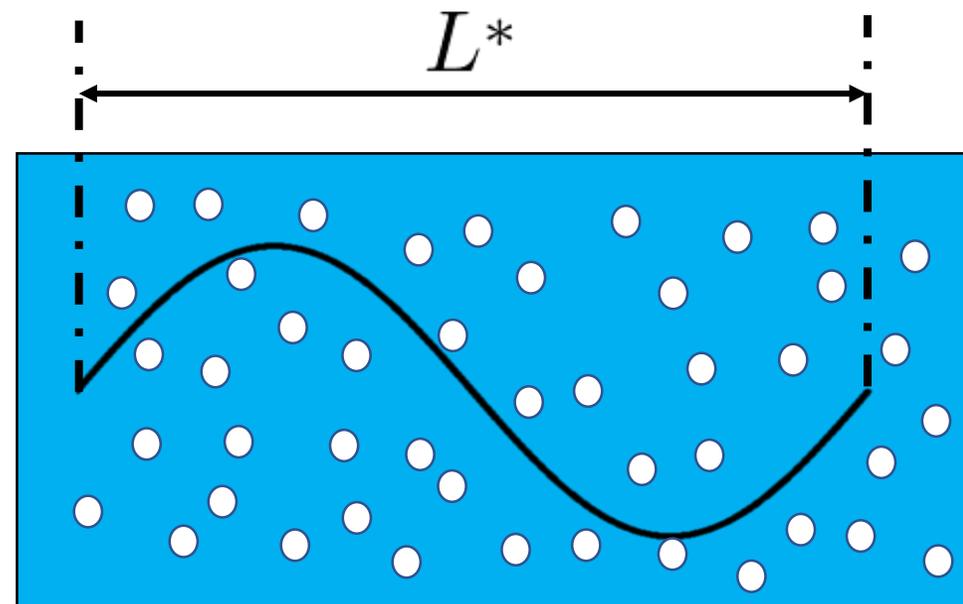
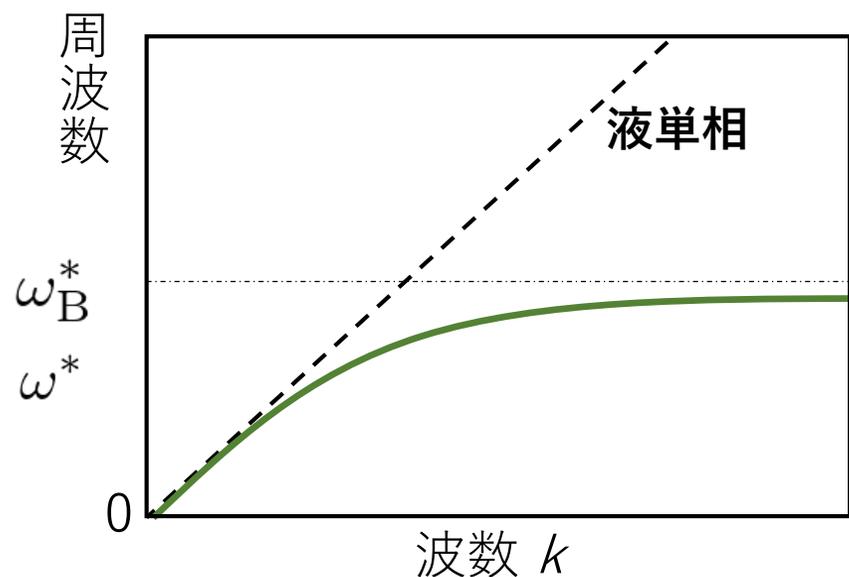
• λ のスケーリング

$$\frac{\lambda^*}{\rho_{L0}^* U^* L^* c_p^*} = \frac{\mu^*}{\rho_{L0}^* U^* L^*} \frac{\lambda^*}{\mu^* c_p^*} = \frac{\mu}{\text{Pr}} \epsilon$$

$\equiv \mu\epsilon$ 常温で約7

具体的なパラメータスケージング

$$\frac{\omega^*}{\omega_B^*} = O(\sqrt{\epsilon}), \quad \frac{U^*}{c_{L0}^*} = O(\sqrt{\epsilon}), \quad \frac{R_0^*}{L^*} = O(\sqrt{\epsilon}) \quad (\epsilon \ll 1)$$



R_1 とその他の変数の関係

$$\alpha_1 = 3(1 - \alpha_0)R_1$$

$$u_1 = -3\alpha_0 R_1$$

$$p_{L1} = -3\alpha_0(1 - \alpha_0)R_1$$

$$T_1 = -3\alpha_0\delta R_1$$

等温変化の時

$$T_i (i = 1, 2 \dots) = 0 \quad \rightarrow \quad \delta = 0$$

線形波動方程式

$$\frac{\partial^2 R_1}{\partial t_0^2} - \frac{\frac{\Delta^2}{\Omega^2} - 3(\gamma - 1)p_{G0}}{3\alpha_0(1 - \alpha_0 - \delta p_{G0})} \frac{\partial^2 R_1}{\partial x_0^2} = 0$$

線形化された基礎方程式系

$$\frac{\partial \alpha_1}{\partial t_0} - 3 \frac{\partial R_1}{\partial t_0} + \frac{\partial u_1}{\partial x_0} = 0$$

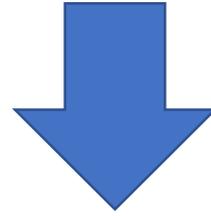
$$\alpha_0 \frac{\partial \alpha_1}{\partial t_0} - (1 - \alpha_0) \frac{\partial u_1}{\partial x_0} = 0$$

$$(1 - \alpha_0) \frac{\partial u_1}{\partial t_0} + \frac{\partial p_{L1}}{\partial x_0} = 0$$

$$\left[3(\gamma - 1)p_{G0} - \frac{\Delta^2}{\Omega^2} \right] R_1 + p_{G0}T_1 - p_{L1} = 0$$

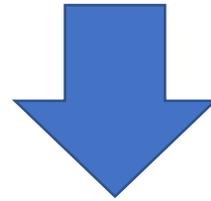
$$(1 - \alpha_0) \frac{\partial T_1}{\partial t_0} - \delta \frac{\partial p_{L1}}{\partial t_0} = 0$$

$$K = \frac{1}{3\alpha_0(1-\alpha_0)} \left[-(1-\alpha_0)\alpha_0 \frac{\partial^2 K_1}{\partial t_0^2} + (1-\alpha_0) \frac{\partial^2 K_2}{\partial t_0^2} + \frac{\partial^2 K_3}{\partial t_0 \partial x_0} \right. \\ \left. + \frac{1-\alpha_0}{1-\alpha_0-\delta p_{G0}} \frac{\partial^3 K_4}{\partial t_0 \partial x_0^2} - \frac{p_{G0}}{1-\alpha_0-\delta p_{G0}} \frac{\partial^2 K_5}{\partial x_0^2} \right]$$



$$\varphi_0 = x_0 - t_0 \Rightarrow \frac{\partial}{\partial t_0} = -\frac{\partial}{\partial \varphi_0}, \quad \frac{\partial}{\partial x_0} = \frac{\partial}{\partial \varphi_0}$$

$$K = \frac{\partial^2}{\partial \varphi_0^2} \left\{ \frac{1}{3\alpha_0(1-\alpha_0)} \left[-(1-\alpha_0)\alpha_0 K_1 + (1-\alpha_0)K_2 - K_3 - \frac{1-\alpha_0}{1-\alpha_0-\delta p_{G0}} \frac{\partial K_4}{\partial \varphi_0} - \frac{p_{G0}}{1-\alpha_0-\delta p_{G0}} K_5 \right] \right\}$$



$$\left(\frac{\partial f}{\partial t_1} + \frac{\partial f}{\partial x_1} \right) + \Pi_0 \frac{\partial f}{\partial \varphi_0} + \Pi_1 f \frac{\partial f}{\partial \varphi_0} + \Pi_2 \frac{\partial^2 f}{\partial \varphi_0^2} + \Pi_3 \frac{\partial^3 f}{\partial \varphi_0^3} = 0$$

$$\Downarrow t_m = \epsilon^m t, \quad x_m = \epsilon^m x$$

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x} + \epsilon \left(\Pi_0 \frac{\partial f}{\partial x} + \Pi_1 f \frac{\partial f}{\partial x} + \Pi_2 \frac{\partial^2 f}{\partial x^2} + \Pi_3 \frac{\partial^3 f}{\partial x^3} \right) = 0$$

$$\Downarrow \tau = \epsilon t, \quad \xi = x - (1 + \epsilon \Pi_0)t$$

$$\frac{\partial R_1}{\partial \tau} + \Pi_1 R_1 \frac{\partial R_1}{\partial \xi} + \Pi_2 \frac{\partial^2 R_1}{\partial \xi^2} + \Pi_3 \frac{\partial^3 R_1}{\partial \xi^3} = 0$$

等温変化の時，本解析と先行研究が一致

• 等温変化

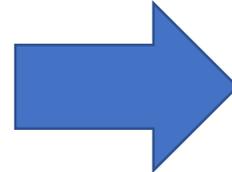
$$\delta = 0$$

$$\gamma = 1$$

• 例：音速

➤ 本解析

$$\frac{\frac{\Delta^2}{\Omega^2} - 3(\gamma - 1)p_{G0}}{3\alpha_0(1 - \alpha_0 - \delta p_{G0})}$$

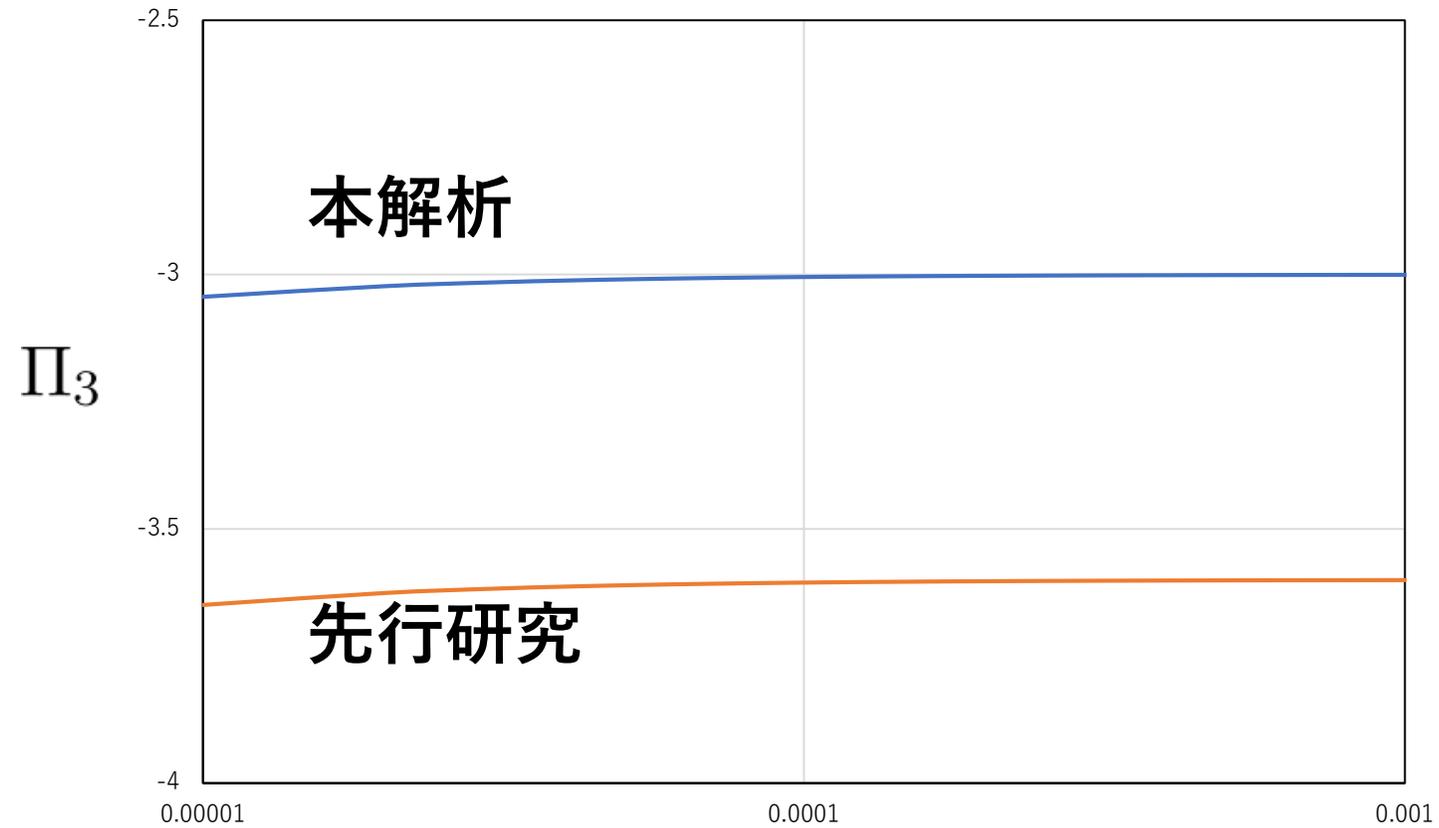


$$\frac{\Delta^2 / \Omega^2}{3\alpha_0(1 - \alpha_0)}$$

➤ 先行研究

$$\frac{\Delta^2 / \Omega^2}{3\alpha_0(1 - \alpha_0)}$$

非線形係数



分散係数

