Weakly Nonlinear Formulation on Acoustic Waves in Liquids Containing Many Spherical Gas Bubbles (110)

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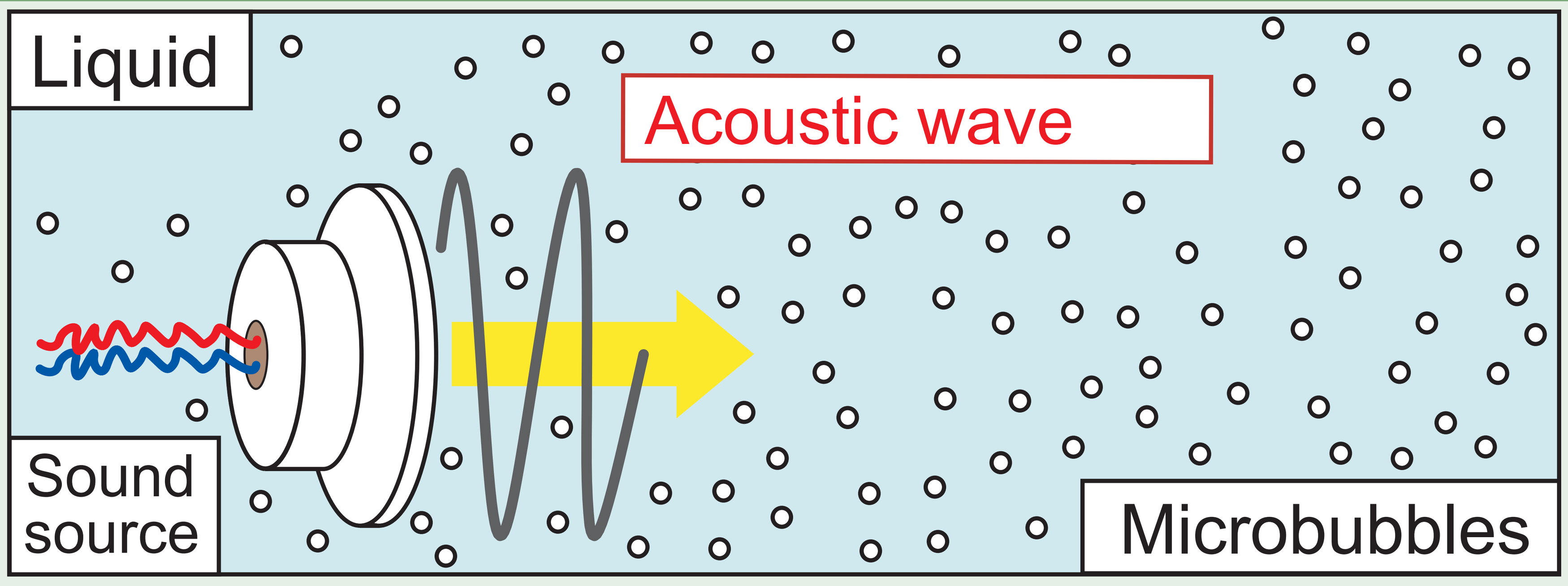
Acoustic waves in bubbly liquids



- Appearance of dispersion due to bubble oscillations ⇒ Competition with weak nonlinearity as a result of long-range propagation at a far field (≫ wavelength)
- Acoustic soliton formation (Weakly nonlinear wave)
- Analogy of water waves as Nonlinear Dispersive waves:
- K–dV Eq. (Shallow; wavelength ≫ depth; weak dispersion)

$$\frac{\partial f}{\partial \tau} + f\frac{\partial f}{\partial \xi} + C_0\frac{\partial^3 f}{\partial \xi^3} = 0$$

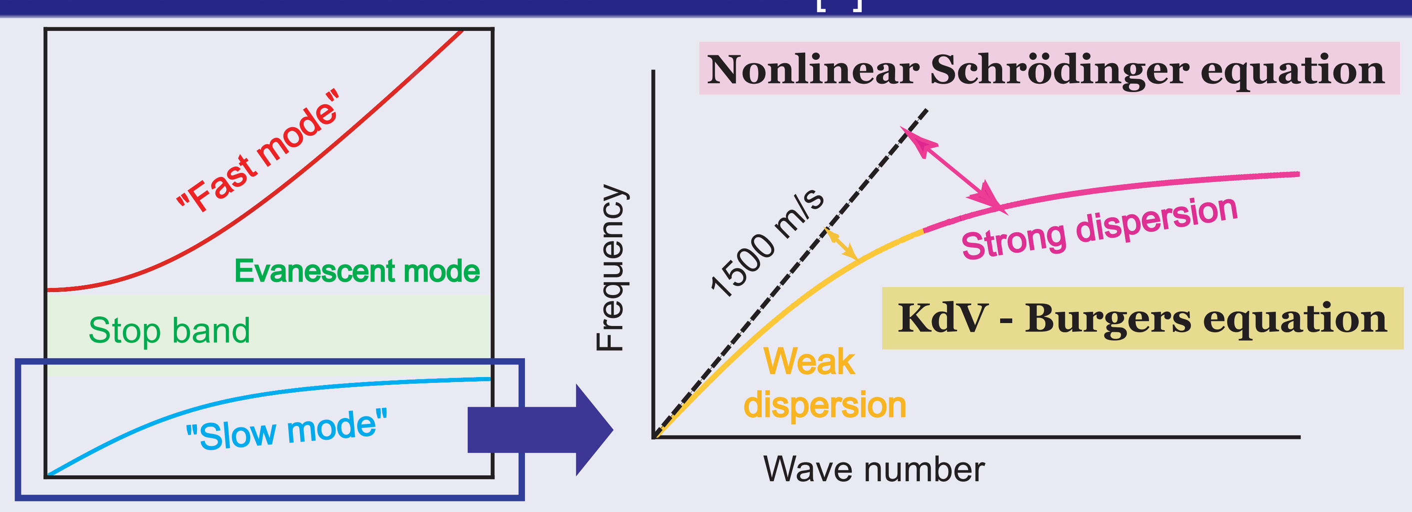
- Non-Linear Schrödinger (NLS) Eq. (Deep; wavelength ≈ depth; strong dispersion)

$$i\frac{\partial A}{\partial \tau} + \frac{d^2\omega}{dk^2}\frac{\partial A}{\partial \xi} + c_0|A|^2 A = 0 \quad (A : \text{complex amplitude})$$

Main Assumptions

- Initially quiescent liquid uniformly containing many spherical gas bubbles ⇒ Volume averaged equations ⇒ Large number of bubbles in averaged volume
- Long-range propagation of 1D progressive waves ⇒ Derivation of Far-field Eqs.
- Liquid compresiblity ⇒ Dispersion and dissipation from oscillating bubbles
- Heat conduction and gas viscosity are dismissed

Two types of Nonlinear evolutions at far fields [1]



- Uniqueness choice of size of set of three dimensionless parameters in terms of Speed, Length & Time
- Magnitudes of Dissipation & Dispersion versus Nonlinearity (≈ Mach number ≈ Amplitude $\epsilon \ll 1$)

$$\left(\frac{U^*}{c_{L0}^*}, \frac{R_0^*}{L^*}, \frac{\omega^*}{\omega_B^*}\right) \equiv \begin{cases} \left(O\left(\sqrt{\epsilon}\right), O\left(\sqrt{\epsilon}\right), O\left(\sqrt{\epsilon}\right)\right), & \text{(for KdVB)} \\ \left(O\left(\epsilon^2\right), O(1), O(1)\right), & \text{(for NLS)} \end{cases}$$

A unified theory to govern various far-fields [1]

♣ Various behaviors at various far fields ⇒ Various Competitions of Dispersion & Dissipation with Nonlinearity

♣ Method of multiple scales with parameter scaling

- Power series expansions of dependent variables in ϵ with respect to $t_m = \epsilon^m t$ and $x_m = \epsilon^m x$
- Relative Sizes of Dissipation and Dispersion to Nonlinearity by using amplitude $\epsilon\,(\ll 1)$,

$$\left(\frac{U^*}{c_{L0}^*}, \frac{R_0^*}{L^*}, \frac{\omega^*}{\omega_B^*}\right) = (O(A), O(B), O(C)) \equiv (O(\epsilon^a), O(\epsilon^b), O(\epsilon^c)).$$

- Substitution above into a generic set of bubbly flows
- Speed U^*/c_{L0}^*: Propagation speed/sound speed in Liquid ⇒ Dissipation due to acoustic radiation
- Length R_0^*/L^*: Bubble radius/Wavelength ⇒ Dispersion due to bubble oscillations
- Time ω^*/ω_B^*: Incident freq./eigenfreq. of single bubble

Basic equations for bubbly flows in a two-fluid model

Conservation laws: Mass & momentum conservation for gas & liquid phases [2]

$$\frac{\partial}{\partial t^*}(\alpha\rho_G^*) + \frac{\partial}{\partial x^*}(\alpha\rho_G^* u_G^*) = 0,$$
$$\frac{\partial}{\partial t^*}[(1-\alpha)\rho_L^*] + \frac{\partial}{\partial x^*}[(1-\alpha)\rho_L^* u_L^*] = 0,$$
$$\frac{\partial}{\partial t^*}(\alpha\rho_G^* u_G^*) + \frac{\partial}{\partial x^*}\left(\alpha\rho_G^* {u_G^*}^2\right) + \alpha\frac{\partial p_G^*}{\partial x^*} = F^*,$$
$$\frac{\partial}{\partial t^*}\left[(1-\alpha)\rho_L^* u_L^*\right] + \frac{\partial}{\partial x^*}\left[(1-\alpha)\rho_L^* {u_L^*}^2\right] + (1-\alpha)\frac{\partial p_L^*}{\partial x^*} + P^*\frac{\partial \alpha}{\partial x^*} = -F^*$$

- α: Volume fraction of gas phase (void fraction)
- p_G^* & p_L^*: Volume-averaged pressure of gas & liquid
- P^*: Surface-averaged liquid pressure at bubble–liquid interface
- F^*: Constitutive: Virtual mass force [3]

$$F^* = -\beta_1\alpha\rho_L^*\left(\frac{D_G u_G^*}{Dt^*} - \frac{D_L u_L^*}{Dt^*}\right) - \beta_2\rho_L^*(u_G^* - u_L^*)\frac{D_G\alpha}{Dt^*} - \beta_3\alpha(u_G^* - u_L^*)\frac{D_G p_L^*}{Dt^*}$$

♣ Bubble dynamics: Keller's equation for radial oscillations of represtntive bubble in compressible liquids

$$\left(1 - \frac{1}{c_{L0}^*}\frac{D_G R^*}{Dt^*}\right)R^*\frac{D_G^2 R^*}{Dt^{*2}} + \frac{3}{2}\left(1 - \frac{1}{3c_{L0}^*}\frac{D_G R^*}{Dt^*}\right)\left(\frac{D_G R^*}{Dt^*}\right)^2$$
$$= \left(1 + \frac{1}{c_{L0}^*}\frac{D_G R^*}{Dt^*}\right)\frac{P^*}{\rho_{L0}^*} + \frac{R^*}{\rho_{L0}^* c_{L0}^*}\frac{D_G}{Dt^*}(p_L^* + P^*)$$

- Acoustic radiation ⟹ Attenuation of oscillations and Waves

Above set of equations is closed by

$$\frac{p_G^*}{p_{G0}^*} = \left(\frac{\rho_G^*}{\rho_{G0}^*}\right)^\gamma, \quad p_L^* = p_{L0}^* + \frac{\rho_{L0}^* c_{L0}^{*2}}{n}\left[\left(\frac{\rho_L^*}{\rho_{L0}^*}\right)^n - 1\right],$$
$$\frac{\rho_G^*}{\rho_{G0}^*} = \left(\frac{R_0^*}{R^*}\right)^3, \quad p_G^* - (p_L^* + P^*) = \frac{2\sigma^*}{R^*} + \frac{4\mu^*}{R^*}\frac{D_G R^*}{Dt^*}$$

Singular perturbation analysis (e.g. Jeffery & Kawahara, 1982)

Perturbation expansions of dependent variables:

$$f/f_0 = 1 + \epsilon f_1 + \epsilon^2 f_2 + \cdots,$$
$$u_G^*/U^* = \epsilon u_{G1} + \epsilon^2 u_{G2} + \cdots,$$
$$\rho_L^*/\rho_{L0}^* = 1 + \epsilon^\kappa \rho_{L1} + \epsilon^{\kappa+1}\rho_{L2} + \cdots,$$

where $\kappa = 2$ for KdVB and $\kappa = 5$ for NLS ⇒ Uniqueness treatment of ρ_L^* [1]

Result: KdVB and NLS equations

(i) Long wave in low frequency ⇒ KdVB Eq.

$$\frac{\partial f}{\partial \tau} + \Pi_1 f\frac{\partial f}{\partial \xi} + \Pi_2\frac{\partial^2 f}{\partial \xi^2} + \Pi_3\frac{\partial^3 f}{\partial \xi^3} = 0$$

(ii) Envelope of short carrier wave in high frequency ⇒ NLS Eq.

$$\mathrm{i}\frac{\partial A}{\partial \tau} + \nu_3\frac{\partial^2 A}{\partial \xi^2} + \nu_1|A|^2 A + \mathrm{i}\nu_2 A = 0$$

- Competition of Dissipation, Dispersion & Nonlinearity at two types of far fields
- Coefficients Π_i's & ν_i's include α_0, R_0^*, γ, ω^*, ...

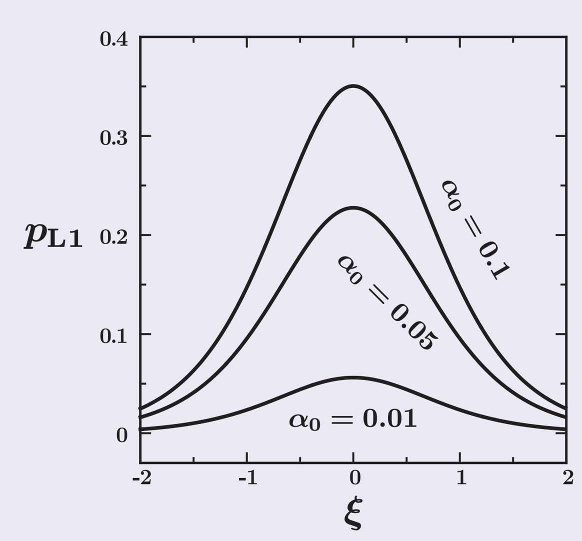
Solitary wave solution of KdV (not KdVB)

A steady travelling wave solution as a particular solution of KdV Eq. without dissipation A soliton solution for liquid pressure:

$$p_{L1} = \mathrm{sech}^2\left(\sqrt{\frac{-\Omega^2/\Delta^2\Pi_1}{12\Pi_3}}\,\xi\right)$$

$$(\Pi_1 < 0,\ \Pi_3 > 0,\ \tau = 0)$$

Height of solitons decreases with decreasing void fraction



Summary

Unified derivation method for nonlinear wave equations in bubbly liquids, which is based on parameter scaling appropriate to specific wave phenomenon, is proposed:

$$\left(\frac{U^*}{c_{L0}^*}, \frac{R_0^*}{L^*}, \frac{\omega^*}{\omega_B^*}\right) \equiv (O(\epsilon^A), O(\epsilon^B), O(\epsilon^C)).$$

[1] Kanagawa et al., *J. Fluid Sci. Technol.*, **5** (2010), 351. [2] Egashira, Yano & Fujikawa, *Fluid Dyn. Res.*, **34** (2004), 317. Yano et al., *J. Phys. Soc. Jpn.*, **75** (2006), 104401. [4] Kanagawa, *J. Acoust. Soc. Am.* **137** (2015), 2642.